On spectral minimal partitions : the case of the sphere

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Abstract

In continuation of [23], [22], [8] and [7], we analyze the properties of spectral minimal partitions and focus in this paper our analysis on the case of the sphere. We prove that a minimal 3-partition for the sphere $\mathbb{S}^2$ is up to rotation the so called **Y**-partition. This question is connected to a celebrated conjecture of Bishop in harmonic analysis.

1 Introduction

Motivated by questions related to some conjecture of Bishop [6], we continue the analysis of spectral minimal partitions developed for plane domains in [23], [22], [8], [7] and analyze the case of the two-dimensional sphere $\mathbb{S}^2$.

In the whole paper the Laplacian is the Laplace-Beltrami operator on $\mathbb{S}^2$. We describe as usual $\mathbb{S}^2$ in $\mathbb{R}^3_{x,y,z}$ by the spherical coordinates,

$$x = \cos\phi \sin\theta, y = \sin\phi\sin\theta, z = \cos\theta\,, \text{ with } \phi \in [-\pi,\pi[\,,\ \theta \in]0,\pi[\,, \quad (1.1)$$

and we add the two poles "North" and "South", corresponding to the two points $(0,0,1)$ and $(0,0,-1)$.
If Ω is a regular bounded open set with piecewise $C^{1,+}$ boundary[1], we consider the Dirichlet Laplacian $H = H(\Omega)$ and we would like to analyze the question of the existence and of the properties of minimal partitions for open sets Ω on $\mathbb{S}^2$. When Ω is strictly contained in $\mathbb{S}^2$, the question is not fundamentally different of the case of planar domains, hence we will focus on

[1] i.e. with piecewise $C^{1,\alpha}$ boundary, for some $\alpha > 0$

the case of the whole sphere $\mathbb{S}^2$ and on the search for possible candidates for minimal k-partitions of the sphere for k small.

To be more precise, let us now recall a few definitions that the reader can for example find in [23]. For $1 \leq k \in \mathbb{N}$ and $\Omega \subset \mathbb{S}^2$, we call a **spectral k-partition**[2] of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of pairwise disjoint open regular domains such that

$$\cup_{i=1}^k D_i \subset \Omega. \tag{1.2}$$

It is called **strong** if

$$\mathrm{Int}\,(\overline{\cup_{i=1}^k D_i}) \setminus \partial\Omega = \Omega\,. \tag{1.3}$$

We denote by $\mathfrak{O}_k$ the set of such partitions. For $\mathcal{D} \in \mathfrak{O}_k$ we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i)\,, \tag{1.4}$$

where $\lambda(D_i)$ is the ground state energy of $H(D_i)$, and

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda(\mathcal{D})\,. \tag{1.5}$$

We call a spectral minimal k-partition, a k-partition $\mathcal{D} \in \mathfrak{O}_k$ such that

$$\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})\,.$$

More generally we can consider (see in [23]) for $p \in [1, +\infty[$

$$\Lambda^p(\mathcal{D}) = (\frac{1}{k}\sum_i \lambda(D_i)^p)^{\frac{1}{p}}\,, \tag{1.6}$$

and

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda^p(\mathcal{D})\,. \tag{1.7}$$

We write $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$ and recall the monotonicity property

$$\mathfrak{L}_{k,p}(\Omega) \leq \mathfrak{L}_{k,q}(\Omega) \text{ if } p \leq q\,. \tag{1.8}$$

The notion of p-minimal k-partition can be extended accordingly, by minimizing $\Lambda^p(\mathcal{D})$.
We would like to give in this article the proof of the following theorem.

[2] We say more shortly "k-partition".

Theorem 1.1
Any minimal 3*-partition of* $\mathbb{S}^2$ *is up to a fixed rotation obtained by the so called* $\mathbf{Y}$*-partition whose boundary is given by the intersection of* $\mathbb{S}^2$ *with the three half-planes defined respectively by* $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$ *. Hence*

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,. \tag{1.9}$$

This theorem is immediately related (actually a consequence of) to a conjecture of Bishop (Conjecture 6) proposed in [6] stating that :

Conjecture 1.2 (Bishop 1992)
The minimal 3*-partition for* $\frac{1}{3}(\sum_{i=1}^{3}\lambda(D_i))$ *corresponds to the* $\mathbf{Y}$*-partition.*

We can indeed observe that if for some (k,p) there exists a p-minimal k-partition $\mathcal{D}_{k,p}$ such that $\lambda(D_i) = \lambda(D_j)$ for all i,j, then by the monotonicity property $\mathcal{D}_{k,p}$ is a q-minimal partition for any $q \geq p$.

Remark 1.3
At the origin, Bishop's Conjecture was motivated by the analysis of the properties of Harmonic functions in conic sets. The whole paper by Friedland-Hayman [19] (see also references therein) which inspires our Section 7 is written in this context. The link between our problem of minimal partitions and the problem in harmonic analysis can be summarized in this way. If we consider a homogeneous Lipschitzian function of the form $u(x) = r^\alpha g(\theta,\phi)$ *in* $\mathbb{R}^3$*, which is harmonic outside its nodal set and such that the complementary of the nodal set divides the sphere in three parts, then*

$$\alpha(\alpha+1) \geq \mathfrak{L}_3(\mathbb{S}^2)\,.$$

Hence Theorem 1.1 (and more specifically (1.9)*) implies* $\alpha \geq 3/2$*. This kind of property can be useful to improve some statements in [12, 13] (see inside the proofs of Lemmas 2 in [12] and 4.1 in [13]).*

A similar question was analyzed (with partial success) when looking in [22] at candidates of minimal 3-partitions of the unit disk $D(0,1)$ in $\mathbb{R}^2$. The most natural candidate was indeed the Mercedes Star, which is the 3-partition given by three disjoint sectors with opening angle $2\pi/3$, i.e.

$$D_1 = \{x \in \Omega \mid \omega \in]0, 2\pi/3[\} \tag{1.10}$$

and D_2, D_3 are obtained by rotating D_1 by $2\pi/3$, respectively by $4\pi/3$. Hence the Mercedes star in [22] is replaced here by the $\mathbf{Y}$-partition in Theorem 1.1. We observe that $\mathbf{Y}$-partition can also be described the inverse image of the mercedes-star partition by the map $\mathbb{S}^2 \ni (x, y, z) \mapsto (x, y) \in D(0, 1)$.

Here let us mention the two main statements giving the proof of Theorem 1.1.

Proposition 1.4
If $\mathcal{D} = (D_1, D_2, D_3)$ is a 3-minimal partition, then its boundary contains two antipodal points.

The proof of Proposition 1.4 will be achieved in Section 5 and involves Euler's formula and the thorem of Lyusternik and Shnirelman.

Proposition 1.5
If there exists a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ of $\mathbb{S}^2$ with two antipodal points in $\cup_i \partial D_i$, then it is (after possibly a rotation) the $\mathbf{Y}$-partition.

The proof of Proposition 1.5 will be done in Section 6 by lifting this 3-partition on the double covering $\mathbb{S}^2_{\mathcal{C}}$ of $\ddot{\mathbb{S}}^2$, where $\ddot{\mathbb{S}}^2$ is the sphere minus two antipodal points.

More precisely, following what has been done in the approach of the Mercedes-star conjecture in [22], the steps for the proof of Theorem 1.1 (or towards Conjecture 1.2 if we were able to show that the minimal partition for Λ^1 has all the $\lambda(D_j)$ equal) are the following :

1. One has to prove that minimal partitions on $\mathbb{S}^2$ exist and share the same properties as for planar domains : regularity and equal angle meeting property. This will be done in Section 2.
2. One can observe that the minimal 3-partition cannot be a nodal partition. This is a consequence of Theorem 2.8 in Section 2 and of the fact that the multiplicity of the second eigenvalue (i.e. the first non zero one) is more than 2 actually 3.
3. The Euler formula implies that there exists only one possible type of minimal 3-partitions. Its boundary consists of two points x_1 and x_2 and three arcs joining these two points. This will be deduced in Subsection 4.1.

4. The next point is to show a minimal partition has in its boundary two antipodal points.

5. The next point is that any minimal 3-partition which contains two antipodal points in its boundary can be lifted in a symmetric 6-partition on the double covering $\mathbb{S}^2_{\mathcal{C}}$. More precisely, if $\mathcal{D} = (D_1, D_2, D_3)$ and Π is the canonical projection of $\mathbb{S}^2_{\mathcal{C}}$ onto $\ddot{\mathbb{S}}^2$, we get the 6-partition $\mathcal{D}_{\mathcal{C}}$ by considering
$$\mathcal{D}_{\mathcal{C}} = (D_1^+, D_2^+, D_3^+, D_1^-, D_2^-, D_3^-)\,,$$
where for $j = 1, 2, 3$ D_j^+ and D_j^- denote the two components of $\Pi^{-1}(D_j)$. If $\mathcal{I}$ denotes the map on $\mathbb{S}^2_{\mathcal{C}}$ defined, for $m \in \mathbb{S}^2_{\mathcal{C}}$ by
$$\Pi(\mathcal{I}(m)) = \Pi(m) \text{ with } \mathcal{I}(m) \neq m\,, \tag{1.11}$$
we observe that
$$\mathcal{I}(D_j^+) = D_j^- \,.$$

6. The last point is to show that on this double covering a minimal symmetric 6-partition is necessarily the double **Y**-partition, which is the inverse image in $\mathbb{S}^2_{\mathcal{C}}$ of the **Y**-partition.

All these points will be detailed in the following sections together with analogous questions in the case of minimal 4-partitions.

In the last section, we will describe what can be said towards the proof of Bishop's conjecture and about the large k behavior of $\mathfrak{L}_k$ using mainly the tricky estimates of Friedland-Hayman [19].

2 Definitions, notations and extension of previous results to the sphere.

We first recall more notation, definitions and results essentially extracted of [23] but which have to be extended from the case of planar domains to the case of domains in $\mathbb{S}^2$.

Definition 2.1
We say that an open domain $D \subset \mathbb{S}^2$ is ***regular*** *if it satisfies an interior cone condition and if ∂D is the union a finite number of simple regular arcs $\gamma_i(\overline{I_i})$, with $\gamma_i \in \mathcal{C}^{1,+}(\overline{I_i})$ with no mutual nor self intersections, except possibly at the endpoints.*

For a given set $\Omega \subset \mathbb{S}^2$, we are interested in the eigenvalue problem for $H(\Omega)$, the Dirichlet realization of the Laplace Beltrami operator in Ω. We shall denote for any open domain Ω by $\lambda(\Omega)$ the lowest eigenvalue of $H(\Omega)$. We define for any eigenfunction u of $H(\Omega)$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{2.1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of such a function will be called $\mu(u)$.

If $\mathcal{D}$ is a strong partition, we say $D_i, D_j \in \mathcal{D}$ are **neighbors** if

$$\mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega \text{ is connected} \tag{2.2}$$

and write in this case $D_i \sim D_j$. We then define a **graph** $G(\mathcal{D})$ by associating to each $D_i \in \mathcal{D}$ a vertex v_i and to each pair $D_i \sim D_j$ we associate an edge $e_{i,j}$.

Attached to a regular partition $\mathcal{D}$ we can associate its boundary $N = N(\mathcal{D})$ which is the closed set in $\overline{\Omega}$ defined by

$$N(\mathcal{D}) = \overline{\bigcup_i (\partial D_i \cap \Omega)}. \tag{2.3}$$

This leads us to introduce the set $\mathcal{M}(\Omega)$ of the regular closed sets.

Definition 2.2
A closed set $N \subset \overline{\Omega}$ belongs to $\mathcal{M}(\Omega)$ if N meets the following requirements:
(i) *There are finitely many distinct critical points $x_i \in \Omega \cap N$ and associated positive integers $\nu(x_i)$ with $\nu(x_i) \geq 3$ such that, in a sufficiently small neighborhood of each of the x_i, N is the union of $\nu(x_i)$ disjoint (away from x_i non self-crossing) smooth arcs with one end at x_i (and each pair defining at x_i a positive angle in $]0, 2\pi[$) and such that in the complement of these points in Ω, N is locally diffeomorphic to a smooth arc. We denote by $X(N)$ the set of these critical points.*
(ii) *$\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i, such that at each z_i, $\rho(z_i)$, with $\rho(z_i) \geq 1$ arcs hit the boundary. Moreover for each $z_i \in \partial\Omega$, then N is near z_i the union of ρ_i distinct smooth arcs which hit z_i with strictly positive distinct angles. We denote by $Y(N)$ the set of these critical points.*

Conversely, if N is a regular closed set, then the family $\mathcal{D}(N)$ of connected components of $\Omega \setminus N$ belongs (by definition) to $\mathcal{R}(\Omega)$, hence regular and strong.

Definition 2.3
We will say that a closed set has the ***equal angle meeting property*** *(***eamp***), if the arcs meet with equal angles at each critical point* $x_i \in N \cap \Omega$ *and also with equal angles at the* $z_i \in N \cap \partial\Omega$*. For the boundary points* z_i *we mean that the two arcs in the boundary are included.*

We will say that the partition is **eamp**-regular if it is regular and satisfies the equal angle meeting property.

It has been proved by Conti-Terracini-Verzini [14, 15, 16] that

Theorem 2.4
For any k *there exists a minimal* **eamp***-regular strong* k*-partition.*

It has been proved in [23] the

Theorem 2.5
Any minimal spectral k*-partition admits a representative which is* **eamp***-regular and strong.*

A basic result concerns the regularity (up to the boundary if any) of the nodal partition associated to an eigenfunction.

We first observe that the results about minimal partitions for plane domains can be transfered to the sphere $\mathbb{S}^2$. It is indeed enough to use the stereographic projection on the plane which gives an elliptic operator on the plane with analytic coefficients.This map is a conformal map, hence respecting the angles. The regularity questions being local there are no particular problem for recovering the equal angle meeting property.

A natural question is whether a minimal partition is the nodal partition induced by an eigenfunction. The next theorem gives a simple criterion for a partition to be associated to a nodal set. For this we need some additional definitions.

We recall that the graph $G(\mathcal{D})$ is **bipartite** if its vertices can be colored by two colors (two neighbours having different colors). In this case, we say that the partition is **admissible**. We recall that a collection of nodal domains of an eigenfunction is always admissible.

We have now the following converse theorem [23] :

Theorem 2.6
An admissible minimal k-partition is nodal, i.e. associated to the nodal set of an eigenfunction of $H(\Omega)$ corresponding to an eigenvalue equal to $\mathfrak{L}_k(\Omega)$.

This theorem was already obtained for planar domains in [21] by adding a strong a priori regularity and the assumption that Ω is simply connected. Any subpartition of cardinality 2 corresponds indeed to a second eigenvalue and the criterion of pair compatibility (see [21]) can be applied.

A natural question is now to determine how general is the situation described in Theorem 2.6. As for partitions in planar domains, this can only occur in very particular cases when $k > 2$. If $\lambda_k(\Omega)$ denotes the k-th eigenvalue of the Dirichlet realization of the Laplacian in an open set Ω of $\mathbb{S}^2$, the Courant Theorem says :

Theorem 2.7
The number of nodal domains $\mu(u)$ of an eigenfunction u associated with $\lambda_k(\Omega)$ satisfies $\mu(u) \leq k$.

Then we say, as in [23], that u is **Courant-sharp** if $\mu(u) = k$. For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. In general we have

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \,. \tag{2.4}$$

The next result of [23] gives the full picture of the equality cases :

Theorem 2.8
Suppose that $\Omega \subset \mathbb{S}^2$ is regular. If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\lambda_k(\Omega) = \mathfrak{L}_k(\Omega)$, then

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega) \,,$$

and any minimal k-partition is nodal and admits a representative which is the family of nodal domains of some eigenfunction u associated to $\lambda_k(\Omega)$.

This theorem will be quite useful for showing for example that, for $k = 3$ and $k = 4$, a k-minimal partition of $\mathbb{S}^2$ cannot be nodal. This will be further discussed in Section 3 (see Theorem 3.7).

3 Courant's nodal Theorem with inversion symmetry.

We collect here some easy useful observations for the analysis of the sphere. These considerations already appear in [28] but the application to minimal partitions is new. We consider Courant's Nodal Theorem for $H(\Omega)$ where Ω is an open connected set in $\mathbb{S}^2$. Let

$$\mathbb{S}^2 \ni (x, y, z) \mapsto I(x, y, z) = (-x, -y, -z) \tag{3.1}$$

denotes the inversion map and assume that

$$I\Omega = \Omega\,. \tag{3.2}$$

Note that $\Omega = \mathbb{S}^2$ satisfies the condition. These assumptions imply that we can write $H(\Omega)$ as a direct sum

$$H(\Omega) = H_S(\Omega) \bigoplus H_A(\Omega)\,, \tag{3.3}$$

where $H_S(\Omega)$ and $H_A(\Omega)$ are respectively the restrictions of $H(\Omega)$ to the I-symmetric (resp. antisymmetric) L^2-functions in Ω in $D(H(\Omega))$.

For simplicity we just write H_S, H_A. For the spectrum of $H(\Omega)$, σ, we have

$$\sigma = \sigma_S \cup \sigma_A$$

so that $\sigma_S = \{\lambda_k^S\}_{k=1}^\infty$ and analogously $\sigma_A = \{\lambda_k^A\}_{k=1}^\infty$. It is of independent interest to investigate how σ_S and σ_A are related. Obviously we have

$$\lambda_1^S < \lambda_1^A \leq \lambda_2^A \text{ and } \lambda_1^S < \lambda_2^S\,,$$

by standard spectral theory.

In the present situation we can ask the question of a theorem à la Courant separately for the eigenfunctions of H_S and H_A.

First we note the following easy properties :

1. Suppose u is an eigenfunction and that u is either symmetric or antisymmetric. Then $I\,N(u) = N(u)$, i.e. the nodal set is symmetric with respect to inversion.
2. If u^A is an eigenfunction of H_A, then for each nodal domain D_i of u^A, ID_i is a distinct nodal domain of u^A. Hence the nodal domains come in pairs and $\mu(u^A)$ is even.

3. If u^S is a symmetric eigenfunction, then there are two classes of nodal domains :

   - the symmetric domains,
   $$D_{i,S} = ID_{i,S}\,, \tag{3.4}$$
   - the symmetric pairs of domains $D^1_{i,S}, D^2_{i,S}$ so that
   $$ID^1_{i,S} = D^2_{i,S}\,. \tag{3.5}$$

Theorem 3.1
Suppose that Ω satisfies the symmetry assumption (3.2). Then, if (λ_k^A, u^A) is a spectral pair for $H_A(\Omega)$, we have
$$\mu(u^A) \leq 2k\,. \tag{3.6}$$
If (λ_k^S, u^S) is a spectral pair for $H_S(\Omega)$ and if we denote by $\ell(k)$ the number of pairs of nodal domains of u^S satisfying (3.5) and $m(k)$ the number of domains satisfying (3.4), then we have
$$\ell(k) + m(k) \leq k\,, \tag{3.7}$$
and
$$\mu(u^S) \leq k + \ell(k)\,. \tag{3.8}$$

Remark 3.2
Of course the original Courant Theorem holds, but the above result gives additional informations.

Proof.
We just have to mimick the proof of Courant's original theorem. Let us first show (3.6). We can of course add the condition that :
$$\lambda_{k-1}^A < \lambda_k^A\ .$$
Assume for contradiction that, for some u_k^A we have $\mu(u_k^A) > 2k$. To each pair (D_i, ID_i) of nodal domains of u_k^A, we associate the corresponding ground states, so that
$$H(D_i)\phi_i = \lambda_k^A \phi_i,\ \phi_i \in W_0^{1,2}(D_i)\,,$$
and, with $I\phi_i = -\phi_i \circ I$,
$$H(ID_i, V)I\phi_i = \lambda_k^A I\phi_i\,.$$

We use the variational principle in the form domain of H_A. We have

$$\lambda_k^A = \inf_{\varphi^A \perp \mathcal{Q}_A^{k-1}} \frac{\int_\Omega \left(|\nabla \varphi^A|^2\right) d\mu_{\mathbb{S}^2}}{\int_\Omega |\varphi^A|^2 \, d\mu_{\mathbb{S}^2}} \tag{3.9}$$

where $I\varphi^A = -\varphi^A$ and $\varphi^A \in W_0^{1,2}(\Omega)$. Here $\mathcal{Q}_A^{k-1}$ is just the space spanned by the first $(k-1)$ eigenfunctions of H_A. We proceed now as in the proof of Courant's nodal Theorem. Hence, in other words, we have just replaced in this proof the single domains by pairs of domains.

The proof of (3.8) is similar. □

We can also find some immediate consequences concerning the relation between the σ_A and σ_S. Take for instance a spectral pair (u^A, λ_j^A) and assume that $\mu(u^A) = 2k$. Then we can construct from the $2k$ ground states of each connected component k symmetric ones, each one being supported in a symmetric pair of components. By the variational principle, this time for H_S we obtain

$$\lambda_k^S = \inf_{\varphi^S \perp \mathcal{Q}_S^{k-1}} \frac{\int_\Omega |\nabla \varphi^S|^2 \, d\mu_{\mathbb{S}^2}}{\int_\Omega |\varphi^S|^2 d\mu_{\mathbb{S}^2}}. \tag{3.10}$$

This implies :

Proposition 3.3
Any eigenvalue λ^A of H_A, whose corresponding eigenspace contains an eigenfunction with $2k$ nodal domains, satisfies $\lambda^A \geq \lambda_k^S$.

A similar argument can be also made for the symmetric case if $\ell(k) > 0$. This gives us new versions of Courant-sharp properties.
If we call **pair symmetric partition** a partition which is invariant by the symmetry but such that no element of the partition is invariant, we have the following Courant-sharp analog :

Theorem 3.4
If, for some eigenvalue λ_k^A of $H_A(\Omega)$, there exists an eigenfunction u^A such that $\mu(u^A) = 2k$, then the corresponding family of nodal domains is a minimal pair symmetric partition.

Note that, if the labelling of the eigenvalue (counted as eigenvalue of $H(\Omega)$) is $> 2k$), then it is not a minimal $(2k)$-partition of Ω.

Remark 3.5
Let us finally mention as connected result (see for example [5]), that if Ω *satisfies* (3.2)*, then* $\lambda_2(\Omega) = \lambda_1^A(\Omega)$*.*

Application
It is known that the eigenfunctions are the restriction to $\mathbb{S}^2$ of the homogeneous harmonic polynomials. Moreover, the eigenvalues are $\ell(\ell+1)$ ($\ell \geq 0$) with multiplicity $(2\ell+1)$. Then, the Courant nodal Theorem says that for a spherical harmonic u_ℓ corresponding to $\ell(\ell+1)$, one should have

$$\mu(u_\ell) \leq \ell^2 + 1\,. \tag{3.11}$$

As observed in [28], one can, using the fact that

$$u_\ell(-x) = (-1)^\ell u_\ell(x)\,, \tag{3.12}$$

improve this result by using a variant of Courant's nodal Theorem with symmetry (see Theorem 3.1) and this leads to the improvement

$$\mu(u_\ell) \leq \ell(\ell-1) + 2\,. \tag{3.13}$$

Let us briefly sketch the proof of (3.13). If ℓ is odd, any eigenfunction is odd with respect to inversion. Hence the number of nodal domains is even $\mu(u_\ell) = 2n_\ell$ and there are no nodal domains invariant by inversion. Using Courant's nodal Theorem for H_A, we get with $\ell = 2p+1$ that

$$\begin{array}{ll} n_\ell & \leq \sum_{q=0}^{p-1}(2(2q+1)+1) \quad +1 \\ & = 2p(p-1) + 3p + 1 \\ & = p(2p+1) + 1 \\ & = \frac{1}{2}\ell(\ell-1) + 1\,. \end{array}$$

If ℓ is even, we can only write

$$\mu(u_\ell) = 2n_\ell + p_\ell\,,$$

where p_ℓ is the cardinality of the nodal domains which are invariant by inversion.
Using Courant's nodal Theorem for H_S, we get with $\ell = 2p$, that

$$\begin{array}{ll} n_\ell + p_\ell & \leq \left(\sum_{q=0}^{p-1}(4q+1)\right) + 1 \\ & = 2p(p-1) + p + 1 \\ & = p(2p-1) + 1 \\ & = \frac{1}{2}\ell(\ell-1) + 1\,. \end{array}$$

Using this improved estimate, we immediately obtain :

Proposition 3.6
The only cases where u_ℓ can be Courant-sharp are for $\ell = 0$ and $\ell = 1$.

This proposition has the following consequence :

Theorem 3.7
A minimal k-partition of $\mathbb{S}^2$ is nodal if and only if $k \leq 2$.

Note that in [28, 29] the more sophisticated conjecture (verified for $\ell \leq 6$) is proposed :

Conjecture 3.8

$$\mu(u_\ell) \leq \begin{cases} \frac{1}{2}(\ell+1)^2 & \text{if } \ell \text{ is odd}, \\ \frac{1}{2}\ell(\ell+2) & \text{if } \ell \text{ is even}. \end{cases}$$

Remark 3.9
As indicated by D. Jakobson to one of us, there is also a probabilistic version of this conjecture [31]. V. N. Karpushkin [27] has also the following bound for the number of components :

$$\mu(u_\ell) \leq \begin{cases} (\ell-1)^2 + 2 & \text{if } \ell \text{ is odd}, \\ (\ell-1)^2 + 1 & \text{if } \ell \text{ is even}. \end{cases}$$

This is for ℓ large slightly better than what we obtained with the refined Courant-sharp Theorem. Let us also mention the recent paper [17] and references therein.

Remark 3.10
Considering the Laplacian on the double covering $\mathbb{S}^2_{\mathcal{C}}$ of $\ddot{\mathbb{S}}^2 := \mathbb{S}^2 \setminus \{North, South\}$, Theorems 3.1 and 3.4 hold true where I is replaced by $\mathcal{I}$ (introduced in (1.11)) corresponding to the map $\phi \mapsto \phi + 2\pi$. The $\mathcal{I}$-symmetric eigenfunctions can be identified to the eigenfunctions of $H(\mathbb{S}^2)$ by $u_S(x) = u(\pi(x))$ and the restriction $H_{\mathcal{A}}(\mathbb{S}^2_{\mathcal{C}})$ of $H(\mathbb{S}^2_{\mathcal{C}})$ to the $\mathcal{I}$-antisymmetric space leads to a new spectrum, which will be analyzed in Section 6.

4 On topological properties of minimal 3-partitions of the sphere

As in the case of planar domains, a classification of the possible types of minimal partitions could simplify the analysis. The case of the whole sphere

$\mathbb{S}^2$ shows some difference with for example the case of the disk.

4.1 Around Euler's formula

As in the case of domains in the plane [22], we will use the following result.

Proposition 4.1
Let Ω an open set in $\mathbb{S}^2$ with piecewise $C^{1,+}$ boundary and let $N \in \mathcal{M}(\Omega)$ such that the associate $\mathcal{D}$ consists of μ domains $D_1, \dots, D_\mu$. Let b_0 be the number of components of $\partial\Omega$ and b_1 be the number of components of $N \cup \partial\Omega$. Denote by $\nu(x_i)$ and $\rho(z_i)$ the numbers associated to the $x_i \in X(N)$, respectively $z_i \in Y(N)$. Then

$$\mu = b_1 - b_0 + \sum_{x_i \in X(N)} \left(\frac{\nu(x_i)}{2} - 1\right) + \frac{1}{2} \sum_{z_i \in Y(N)} \rho(z_i) + 1\,. \tag{4.1}$$

Remark 4.2
In the case when $\Omega = \mathbb{S}^2$, the statement simply reads

$$\mu = b_1 + \sum_{x_i \in X(N)} \left(\frac{\nu(x_i)}{2} - 1\right) \; + 1\,, \tag{4.2}$$

where b_1 is the number of components of N.

4.2 Application to 3- and 4-partitions.

The case of 3-partitions
Let us analyze in this spirit the topology of minimal 3-partitions of $\mathbb{S}^2$.
First we recall that a minimal 3-partition cannot be nodal. The multiplicity of the second eigenvalue of $-\Delta_{\mathbb{S}^2}$ is indeed 3. Hence, our minimal 3-partition cannot be admissible.
Let us look now to the information given by Euler's formula. We argue like in [22] for the case of the disk. We recall that at any critical point x_c of N,

$$\nu(x_c) \geq 3\,. \tag{4.3}$$

Hence (4.2) implies that $b_1 \leq 2$ and we have

$$1 \leq b_1 \leq 2\,. \tag{4.4}$$

When $b_1 = 1$, we get as unique solution $\#X(N) = 2$. So $X(N)$ consists of two points x_1 and x_2 such that $\nu(x_i) = 3$ for $i = 1$ and 2. The other case when $\#X(N) = 1$ leads indeed to an even number of half lines arriving to the unique critical point and to an admissible (hence excluded) partition. When $b_1 = 2$, we obtain that $X(N)$ is empty and the partition should be admissible which is excluded. Hence we have shown

Proposition 4.3
If $\mathcal{D}$ is a regular non admissible strong 3-partition of $\mathbb{S}^2$, then $X(N)$ consists of two points x_1 and x_2 such that $\nu(x_i) = 3$, and N consists of three non crossing (except at their ends) arcs joining the two points x_1 and x_2.

In particular this can be applied to minimal 3-partitions of $\mathbb{S}^2$.

The case of 4-partitions

We can analyze in the same way the case of non admissible 4-partitions. Euler's formula leads to the following classification.

Proposition 4.4
If $\mathcal{D}$ is a regular non admissible strong 4-partition of $\mathbb{S}^2$, then we are in one of the following cases :

- *$X(N)$ consists of four points x_i ($i = 1, \ldots, 4$) such that $\nu(x_i) = 3$, and N consists of six non crossing (except at their ends) segments, each one joining two points x_i and x_j ($i \neq j$).*
- *$X(N)$ consists of three points x_i ($i = 1, 2, 3$) such that $\nu(x_1) = \nu(x_2) = 3$, $\nu(x_3) = 4$ and N consists of five non crossing (except at their ends) segments joining two critical points.*
- *$X(N)$ consists of two points x_i ($i = 1, 2$) such that $\nu(x_i) = 3$, and N consists of three non crossing (except at their ends) segments joining the two points x_1 and x_2 and of one closed line.*
- *$X(N)$ consists of two points x_i ($i = 1, 2$) such that $\nu(x_1) = 3$, $\nu(x_2) = 5$ and N consists of four non crossing (except at their ends) segments joining the two critical points and of one non crossing (except at his ends) segment starting from one critical point and coming back to the same one.*

Note that the spherical tetrahedron corresponds to the first type and we recall from Theorem 3.7 that minimal 4-partitions are not admissible.

5 Lyustenik-Shnirelman Theorem and proof of Proposition 1.4

As we have shown in the previous section $N(\mathcal{D})$ consists of two points $x_1 \neq x_2$ and 3 mutually non-crossing arcs $\gamma_1, \gamma_2, \gamma_3$ connecting x_1 and x_2. This means that each D_i has a boundary which is a closed curve which is away from x_1, x_2 smooth.

We first recall the well known theorem of Lyusternik and Shnirelman from 1930, that can be found for instance in [30] on page 23. It states the following.

Theorem 5.1
Suppose $S_1, S_2, \ldots, S_d$ are closed subsets of $\mathbb{S}^{d-1}$ such that $\cup_{i=1}^d S_i = \mathbb{S}^{d-1}$. Then there is at least one S_i that contains a pair of antipodal points.

We will use this theorem in the case $d = 3$ and apply it with S_1, S_2, S_3 defined by

$$S_i = \overline{D}_i. \tag{5.1}$$

In order to prove Proposition 1.4 it suffices to show that

$$N(\mathcal{D}) \cap I\, N(\mathcal{D}) \neq \emptyset\,, \tag{5.2}$$

where we recall that I is the antipodal map.

By Theorem 5.1 we know that there is an $S_i = \overline{D}_i$ which contains a pair of antipodal points. After relabelling the D_i's, we can assume that

$$I\, \overline{D_1} \cap \overline{D_1} \neq \emptyset\ , \tag{5.3}$$

and the goal is to show that

$$I\, \partial D_1 \cap \partial D_1 \neq \emptyset\ . \tag{5.4}$$

Our D_i's have the properties of 3-minimal partitions established in the previous section. In particular ∂D_1 has one component and is also the boundary of ∂D_{13}, where $D_{13} = \mathrm{Int}\,(\overline{D_2} \cup \overline{D_3})$.

The proof is by contradiction. Let us assume by contradiction that

$$I\, \partial D_1 \cap \partial D_1 = \emptyset\ . \tag{5.5}$$

Then there are two possible cases

Case a: $I\,\partial D_1 \subset D_1$

Case b: $I\,\partial D_1 \cap \overline{D_1} = \emptyset$

Let us start with Case a. Again there are two possibilities.

Case a1: $I\,D_1 \subset\subset D_1$ ($\subset\subset$ means compactly included).

Case a2: $I\,D_2 \subset\subset D_1$

But Case $a1$ is in contradiction with the fact that I is an isometry and Case $a2$ is in contradiction with $\lambda(D_2) = \lambda(D_1)$. Of course we could have taken D_3 instead of D_2.

We now look at Case b. Then $I\,\partial D_1$ is delimiting in $\mathbb{S}^2$ two components and D_1 is compactly supported in one component. One of the component is $I\,D_1$. But (5.3) implies that D_1 is in this last component :

$$D_1 \subset\subset I\,D_1\,.$$

This can not be true for two isometric domains. Hence we have a contradiction with (5.5) in all the cases. This achives the proof of the proposition.

6 The Laplacian on $\mathbb{S}^2_{\mathcal{C}}$

6.1 Spherical harmonics with half integers

These spherical harmonics appear from the beginning of Quantum mechanics in connection with the representation theory [32]. We refer to [18] (Problem 56 (NB2) in the first volume together with Problem 133 in the second volume). We are looking for eigenfunctions of the Friedrichs extension of

$$\mathbf{L}^2 = -\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} \tag{6.1}$$

in $L^2(\sin\theta d\theta\, d\phi)$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}\,. \tag{6.2}$$

The standard spherical harmonics, corresponding to $\ell \geq 0$ are defined, for an integer $m \in \{-\ell,\dots,\ell\}$, by

$$Y_{\ell m}(\theta,\phi) = c_{\ell,m}\exp im\phi\frac{1}{\sin^m\theta}(-\frac{1}{\sin\theta}\frac{d}{d\theta})^{\ell-m}\sin^{2\ell}\theta\,, \tag{6.3}$$

where $c_{\ell,m}$ is an explicit normalization constant.
For future extensions, we prefer to take this as a definition for $m \geq 0$ and then to observe that

$$Y_{\ell,-m} = \hat{c}_{\ell,m}\overline{Y_{\ell,m}}\,. \tag{6.4}$$

For $\ell = 0$, we get $m = 0$ and the constant. For $\ell = 1$, we obtain, for $m = 1$, the function $(\theta, \phi) \mapsto \sin\theta \exp i\phi$ and for $m = -1$, the function $\sin\theta \exp -i\phi$ and for $m = 0$ the function $\cos\theta$, which shows that the multiplicity is 3 for the eigenvalue 2.
Of course concerning nodal sets, we look at the real valued functions $(\theta, \phi) \mapsto \sin\theta\cos\phi$ and $(\theta, \phi) \mapsto \sin\theta\sin\phi$ for $|m| = 1$.

As observed a long time ago, these formulas still define eigenfunctions for pairs (ℓ, m) with ℓ a positive half-integer (and not integer), $m \in \{-\ell, \dots, \ell\}$ and $m - \ell$ integer.
For definiteness, we prefer (in the half-integer case) to only consider the pairs with $\ell > 0$ and $m > 0$ and to complete the set of eigenfunctions by introducing

$$\widehat{Y}_{\ell,m} = \overline{Y_{-\ell,m}} \tag{6.5}$$

These functions are only defined on the double covering $\mathbb{S}^2_{\mathcal{C}}$ of $\ddot{\mathbb{S}}^2 := \mathbb{S}^2 \setminus \{\{\theta = 0\} \cup \{\theta = \pi\}\}$, which can be defined by extending ϕ to the interval $]-2\pi, 2\pi]$.
When restricted to $\phi \in]-\pi, \pi]$, they correspond to the antiperiodic problem with respect to period 2π in the ϕ variable.

To show the completeness it is enough to show that, for given $m > 0$, the orthogonal family (indexed by $\ell \in \{m + \mathbb{N}\}$) of functions $\theta \mapsto \psi_{\ell,m}(\theta) := \frac{1}{\sin^m\theta}(-\frac{1}{\sin\theta}\frac{d}{d\theta})^{\ell-m}\sin^{2\ell}\theta$ span all $L^2(]0,\pi[, \sin\theta\, d\theta)$.
For this, we consider $\chi \in C_0^\infty(]0,\pi[)$ and assume that

$$\int_0^\pi \chi(\theta)\psi_{\ell,m}(\theta)\sin\theta d\theta = 0\,,\ \forall \ell \in \{m + \mathbb{N}\}\,.$$

We would like to deduce that this implies $\chi = 0$. After a change of variable $t = \cos\theta$ and an integration by parts, we obtain that this problem is equivalent to the problem to show that, if

$$\int_{-1}^1 \psi(t)\ [(1-t^2)^\ell]^{(\ell-m)}\ dt = 0\,,\ \forall \ell \in \{m + \mathbb{N}\}\,,$$

then $\psi = 0$.
Observing that the space spanned by the functions $(1-t^2)^{-m}((1-t^2)^\ell)^{(\ell-m)}$

(which are actually polynomials of exact order ℓ) is the space of all polynomials we can conclude the completeness.
Hence we have obtained the

Theorem 6.1
The spectrum of the Laplace Beltrami operator on $\mathbb{S}^2_{\mathcal{C}}$ can be described by the eigenvalues $\mu_\ell = \ell(\ell+1)$ ($\ell \in \mathbb{N}/2$), each eigenvalue being of multiplicity $(2\ell+1)$. Moreover the $Y_{\ell,m}$, as introduced in (6.3), (6.4) *and* (6.5), *define an orthonormal basis of the eigenspace E_{μ_ℓ}.*

In particular, for $\ell = \frac{1}{2}$, we get a basis of two orthogonal real eigenfunctions $\sin\frac{\phi}{2}(\sin\theta)^{\frac{1}{2}}$ and $\cos\frac{\phi}{2}(\sin\theta)^{\frac{1}{2}}$ of the eigenspace associated with $\frac{3}{4}$. For $\ell = \frac{3}{2}$, the multiplicity is 4 and the functions $\sin\frac{3\phi}{2}(\sin\theta)^{\frac{3}{2}}$, $\cos\frac{3\phi}{2}(\sin\theta)^{\frac{3}{2}}$, $\sin\frac{\phi}{2}(\sin\theta)^{\frac{1}{2}}\cos\theta$ and $\cos\frac{\phi}{2}(\sin\theta)^{\frac{1}{2}}\cos\theta$ form a basis of the eigenspace associated with the eigenvalue $\frac{15}{4}$.

6.2 Covering argument and minimal partition

Here we give one part of the proof of Proposition 1.5.

Lemma 6.2
Let us assume that there exists a 3-minimal partition $\mathcal{D} = (D_1, D_2, D_3)$ of $\mathbb{S}^2$ containing two antipodal points in its boundary. Then, considering the associated punctured $\ddot{\mathbb{S}}^2$ and the corresponding double covering $\mathbb{S}^2_{\mathcal{C}}$ and denoting by Π the projection of $\mathbb{S}^2_{\mathcal{C}}$ on $\ddot{\mathbb{S}}^2$, $\Pi^{-1}(D_i)$ consists of two components and $\pi^{-1}(\mathcal{D})$ defines a 6-partition of $\mathbb{S}^2_{\mathcal{C}}$ which is pairwise symmetric.

The only point to observe is that, according to the property of a minimal partition established in Proposition 4.3, the boundary of the partition necessarily contains a “broken” line joining the two antipodal points.
Using the minimax principle, one immediately gets that, **under the assumption of the lemma**,

$$\mathfrak{L}_3(\mathbb{S}^2) \geq \lambda^3_{AS}(\mathbb{S}^2_{\mathcal{C}})\,, \tag{6.6}$$

where λ^3_{AS} is the third eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^2_{\mathcal{C}}$ restricted to the antisymmetric spectrum. $\lambda^3_{AS}(\mathbb{S}^2_{\mathcal{C}})$ will be computed in the next subsection.

6.3 Covering argument and Courant-sharp eigenvalues

In the case of the double covering $\mathbb{S}^2_{\mathcal{C}}$ of $\ddot{\mathbb{S}}^2$, we have seen that we have to add the antisymmetric (or antiperiodic) spectrum (corresponding to the map Π, which writes in spherical coordinates : $\phi \mapsto \phi + 2\pi$). This adds the eigenvalue $\frac{3}{4} = \frac{1}{2}(1+\frac{1}{2})$ with multiplicity 2 and the eigenvalue $\frac{15}{4} = \frac{3}{2}(1+\frac{3}{2})$ with multiplicity 4. Hence $\frac{15}{4}$ is the 7-th eigenvalue of the Laplacian on $\mathbb{S}^2_{\mathcal{C}}$ hence not Courant-sharp, but it is the third antisymmetric eigenvalue

$$\lambda^3_{AS} = \frac{15}{4}\,.$$

Hence, observing that the nodal set of an eigenfunction associated to λ^3_{AS} has six nodal domains which are pairwise symmetric and giving by projection the Y-partition, we immediately obtain that under the assumption of the lemma

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,.$$

But Proposition 1.5 says more. For getting this result, we have to prove the following proposition :

Proposition 6.3
Let $\mathfrak{L}^{AS}_{2\ell}(\mathbb{S}^2_{\mathcal{C}})$ the infimum obtained over the pairwise symmetric (by Π) (2ℓ)-partitions of $\mathbb{S}^2_{\mathcal{C}}$. Then, if

$$\mathfrak{L}^{AS}_{2\ell}(\mathbb{S}^2_{\mathcal{C}}) = \lambda^{\ell}_{AS}\,,$$

then λ^{ℓ}_{AS} is Courant-sharp in the sense of the antisymmetric spectrum and any minimal pairwise symmetric $(2\ell)-$partition is nodal.

The proof is the same as for Theorem 1.17 in [23] and Theorem 2.6 in [22], with the difference that we consider everywhere antisymmetric states.

Applying this proposition for $\ell = 3$, we have the proof of Proposition 1.5.

7 On Bishop's approach for minimal 2-partitions and extensions to strong k-partitions

7.1 Main result for $k = 2$

For 2-partitions, it is immediate to show that the minimal 2-partitions realizing $\mathfrak{L}_2(\mathbb{S}^2)$ are given by the two hemispheres. One is indeed in the Courant-

sharp situation. The case of $\mathfrak{L}_{2,p}(\mathbb{S}^2)$ for $p < \infty$ is more difficult. Bishop has described in [6] how one can show that the minimal 2-partitions realizing $\mathfrak{L}_{2,1}(\mathbb{S}^2)$ are also given by the two hemispheres. It is then easy to see that it implies the property for any $p \in [1, +\infty[$. Hence we obtain :

Theorem 7.1
For any $p \in [1, +\infty]$, $\mathfrak{L}_{2,p}(\mathbb{S}^2)$ is realized by the partition of $\mathbb{S}^2$ by two hemispheres.

The proof is based on two theorems due respectively to Sperner [34] and Friedland-Hayman [19]. We will discuss their proof because it will have some consequences for the analysis of minimal 3 and 4-partitions.

7.2 The lower bounds of Sperner and Friedland-Hayman

For a given domain D on the unit sphere S^{m-1} in $\mathbb{R}^m$, Sperner shows the following theorem, which plays on the sphere the same role as the Faber-Krahn Inequality plays in $\mathbb{R}^m$:

Theorem 7.2
Among all sets $E \subset \mathbb{S}^{m-1}$ with given $(m-1)$-dimensional surface area $\sigma_m S$ on (with σ_m being the area of $\mathbb{S}^{m-1}$), a spherical cap has the smallest characteristic constant.

Here the characteristic constant for a domain D is related to the ground state energy by

$$\lambda(D) = \alpha(D)(\alpha(D) + m - 2)\,, \tag{7.1}$$

with $\alpha(D) \geq 0$.
We introduce for short

$$\alpha(S, m) = \alpha(\mathcal{SC}(\sigma_m S))\,, \tag{7.2}$$

where $\mathcal{SC}(\sigma_m S)$ is a spherical cap of surface area $\sigma_m S$.

This theorem is not sufficient in itself for the problem. The second ingredient[3] is a lower bound of $\alpha(S, m)$ by various convex decreasing functions. It is indeed proven[4] in [19] that :

[3]See for example [2] p. 441 or C. Bishop [6]
[4]We only write the result for $m = 3$ but (7.3) holds for any $m \geq 3$.

Theorem 7.3
We have the following lower bound :

$$\alpha(S,3) \geq \Phi_3(S)\,, \tag{7.3}$$

where Φ_3 *is the convex decreasing function defined by*

$$\Phi_3 = \max(\widehat{\Phi}_3, \Phi_\infty)\,, \tag{7.4}$$

$$\Phi_\infty(S) = \begin{cases} \frac{1}{2}\log(\frac{1}{4S}) + \frac{3}{2}\,, & \textit{if } 0 < S \leq \frac{1}{4}\,, \\ 2(1-S)\,, & \textit{if } \frac{1}{4} \leq S < 1\,. \end{cases} \tag{7.5}$$

$$\widehat{\Phi}_3(S) = \begin{cases} 2(1-S)\,, & \textit{if } \frac{1}{2} \leq S < 1\,, \\ \frac{1}{2}j_0\left(\frac{1}{S} - \frac{1}{2}\right)^{\frac{1}{2}} - \frac{1}{2}\,, & \textit{if } S < \frac{1}{2}\,, \end{cases} \tag{7.6}$$

and j_0 *being the first zero of Bessel's function of order* 0 *:*

$$j_0 \sim 2.4048\,. \tag{7.7}$$

7.3 Bishop's proof for 2-partitions

With these two ingredients, we observe (following a remark of C. Bishop) that, for a 2-partition, we have necessarily

$$\alpha(D_1) + \alpha(D_2) \geq 2\,, \tag{7.8}$$

the equality being obtained for two hemispheres.
The minimization for the sum corresponds to

$$\inf\left(\alpha(D_1)(\alpha(D_1)+1) + \alpha(D_2)(\alpha(D_2)+1)\right)\,. \tag{7.9}$$

This infimum is surely larger or equal to

$$\inf_{\alpha_1+\alpha_2\geq 2\,,\,\alpha_1\geq 0, \alpha_2\geq 0}\left(\alpha_1(\alpha_1+1) + \alpha_2(\alpha_2+1)\right)\,.$$

It is then easy to see that the infimum is obtained for $\alpha_1 = \alpha_2 = 1$,

$$\inf_{\alpha_1+\alpha_2\geq 2\,,\,\alpha_1\geq 0, \alpha_2\geq 0}\left(\alpha_1(\alpha_1+1) + \alpha_2(\alpha_2+1)\right) = 4\,. \tag{7.10}$$

This gives a lower bound for $\mathfrak{L}_{2,1}(\mathbb{S}^2)$ which is equal to the upper bound of $\mathfrak{L}_2(\mathbb{S}^2)$ and which is attained for the two hemispheres. This achieves the proof of Theorem 7.1.

Remark 7.4
A natural question is to determine under which condition the infimum of $\Lambda^1(\mathcal{D})$ for $\mathcal{D} \in \mathfrak{O}_2$ is realized for a pair (D_1, D_2) such that $\lambda(D_1) = \lambda(D_2)$. Let us illustrate the question by a simple example. If we consider two disks C_1 and C_2 such that $\lambda(C_1) < \lambda(C_2) \leq \lambda_2(C_1)$, it is not too difficult to see that if we take Ω as the union of these two disks and of a thin channel joining the two disks, then $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$ will be very close to $\lambda(C_2)$ and the infimum of $\Lambda^1(\mathcal{D})$ will be less than $\frac{1}{2}(\lambda(C_1)+\lambda(C_2))$. Hence we will have strict inequality if the channel is small enough. We refer to [10, 11, 3, 26] for the spectral analysis of this type of situation. These authors are actually more interested in the symmetric situation where tunneling plays an important role.

7.4 Application to general k-partitions

One can also discuss what can be obtained in the same spirit for k-partitions ($k \geq 3$). This will not lead to the proof of Bishop's conjecture but give rather accurate lower bounds corresponding in a slightly different context to the ones proposed in Friedland-Hayman [19] for harmonic functions in cones of $\mathbb{R}^m$.

Let us first mention the easy result extending (7.10).

Lemma 7.5
Let $k \in \mathbb{N}^*$ and $\rho > 0$. If

$$T^{k,\rho} := \{\alpha \in \overline{\mathbb{R}}_+^k \mid \sum_{j=1}^{k} \alpha_j \geq \rho\},$$

then

$$\frac{1}{k} \inf_{\alpha \in T^{k,\rho}} \sum_{j=1}^{k} \alpha_j(\alpha_j + 1) \geq \frac{\rho}{k}(\frac{\rho}{k} + 1)\,.$$

For a k-partition $\mathfrak{D} = (D_1, \ldots, D_k)$, the corresponding characteristic numbers satisfy :

$$\sum_j \alpha(D_j) \geq \sum_j \Phi_3(S_j) \tag{7.11}$$

with

$$\sigma_3\, S_j = \text{ Area } (D_j)\ (j = 1, \ldots, k)\,, \tag{7.12}$$

and

$$\sum_j S_j = 1\,. \tag{7.13}$$

Using the convexity of Φ_3, we obtain

Proposition 7.6
If $\mathfrak{D} = (D_j)_{j=1,\dots,k}$ is a strong k-partition of $\mathbb{S}^2$, then

$$\frac{1}{k}\sum_{j=1}^{k} \alpha(D_j) \geq \Phi_3(\frac{1}{k})\,. \tag{7.14}$$

Applying Lemma 7.5 with $\rho = k\Phi_3(\frac{1}{k})$, this leads together with (7.14) and (1.8) to the lower bound of $\mathfrak{L}_{k,1}(\mathbb{S}^2)$:

Proposition 7.7

$$\mathfrak{L}_k(\mathbb{S}^2) \geq \mathfrak{L}_{k,1}(\mathbb{S}^2) \geq \Phi_3(\frac{1}{k})\,(1+\Phi_3(\frac{1}{k}))\,. \tag{7.15}$$

Let us see what it gives coming back to the definition of Φ_3.

Corollary 7.8

$$\mathfrak{L}_k(\mathbb{S}^2) \geq \mathfrak{L}_{k,1}(\mathbb{S}^2) \geq \gamma_k\,, \tag{7.16}$$

with

$$\gamma_k := \Phi_\infty(\frac{1}{k})\,(1+\Phi_\infty(\frac{1}{k}))\,, \tag{7.17}$$

and

$$\Phi_\infty(\frac{1}{k}) := \begin{cases} \frac{2(k-1)}{k} & \text{if } k \leq 4\,,\\ 2\log(\frac{k}{4}) + \frac{3}{2} & \text{if } k > 4\,. \end{cases} \tag{7.18}$$

In particular

$$\gamma_2 = 2\,,\ \gamma_3 = \frac{28}{9}\,,\ \gamma_4 = \frac{15}{4}\,. \tag{7.19}$$

We note that γ_2 is optimal and that $\gamma_3 < \frac{15}{4}$. Hence for $k = 3$, the lower bound is not optimal and does not lead to a proof of Bishop's Conjecture. Let us now consider the estimates associated with Φ_3.

Corollary 7.9

$$\mathfrak{L}_k(\mathbb{S}^2) \geq \mathfrak{L}_{k,1}(\mathbb{S}^2) \geq \delta_k\,, \tag{7.20}$$

with

$$\delta_k := \widehat{\Phi}_3(\frac{1}{k})\,(1+\widehat{\Phi}_3(\frac{1}{k}))\,, \tag{7.21}$$

In particular
$$\delta_3 = \frac{5}{8} j_0^2 - \frac{1}{4}\,,\ \delta_4 = \frac{7}{8} j_0^2 - \frac{1}{4}\,. \tag{7.22}$$

7.5 Discussion for the cases $k = 3$ and $k = 4$.

We observe that $\delta_3 > \gamma_3$ and $\delta_4 > \gamma_4$. Small computations[5] show indeed that
$$\widehat{\Phi}_3(\frac{1}{3}) \sim 1.401\,, \tag{7.23}$$
which is higher than $\Phi_\infty(\frac{1}{3}) = \frac{4}{3}$, and
$$\widehat{\Phi}_3(\frac{1}{4}) \sim 1.748\,, \tag{7.24}$$
which is higher than $\Phi_\infty(\frac{1}{4}) = \frac{3}{2}$. This leads to the lower bound :

Proposition 7.10
$$\mathfrak{L}_4(\mathbb{S}^2) \geq \mathfrak{L}_{4,1}(\mathbb{S}^2) > 15/4 = \mathfrak{L}_3(\mathbb{S}^2)\,. \tag{7.25}$$

In particular the best lower bound of $\mathfrak{L}_{4,1}(\mathbb{S}^2)$ is approximately
$$\delta_4 \sim 4.8035\,. \tag{7.26}$$

Note that by a third method, one can find in [19] (Theorem 5 and table 1, p. 155, computed by J.G. Wendel) another convex function $\tilde{\Phi}$ such that $\alpha(D) \geq \tilde{\Phi}(S)$ and
$$\tilde{\Phi}(\frac{1}{3}) \sim 1.41167\,. \tag{7.27}$$
Note that $\tilde{\Phi}(\frac{1}{4}) < \widehat{\Phi}_3(\frac{1}{4})$ so this improvment occurs only for 3-partitions.

In the case of $\mathbb{S}^2$, unlike the case of the square or the disk, the minimal 4-partition is not nodal (as proven in Theorem 3.7). Note that this implies that the 4-minimal partition realizing $\mathfrak{L}_{4,p}(\Omega)$ for $p \in [1, +\infty]$ is neither nodal.
As already mentioned in [19], there is at least a natural candidate which is the spherical regular tetrahedron. Numerical computations[6] give, for the corresponding 4-partition $\mathcal{D}_4^{Tetra}$,
$$\Lambda(\mathcal{D}_4^{Tetra}) \sim 5.13\,. \tag{7.28}$$

[5] already done in [19]
[6] transmitted to us by M. Costabel

Hence we obtain that

$$\frac{15}{4} < \mathfrak{L}_4(\mathbb{S}^2) \leq \Lambda(\mathcal{D}_4^{Tetra}) < 6 = L_4(\mathbb{S}^2)\,. \tag{7.29}$$

It is interesting to compare it with (7.26).
According to a personal communication of M. Dauge, one can also observe that the largest circle inside a face of the tetrahedron is actually a nodal line corresponding to an eigenfunction with eigenvalue 6 (up to a rotation, this is the (restriction to $\mathbb{S}^2$ of the) harmonic polynomial $\mathbb{S}^2 \ni (x,y,z) \mapsto x^2+y^2-2z^2$. This gives directly the comparison $\Lambda(\mathcal{D}_4^{Tetra}) < 6$.

7.6 Large k lower bounds

We can push the argument by looking at the asymptotic as $k \to +\infty$ of δ_k. This gives :

$$\mathfrak{L}_{1,k}(\mathbb{S}^2) \geq \frac{1}{4} j_0^2 k - \frac{1}{8}\, j_0^2 - \frac{1}{4}\,. \tag{7.30}$$

We note that at least for k large, this is much better than the trivial lower bound

$$\mathfrak{L}_{1,k}(\mathbb{S}^2) \geq \frac{1}{k}\mathfrak{L}_k(\mathbb{S}^2)\,. \tag{7.31}$$

We discuss below in Remark 7.12 an independent improvment.
Multiplying (7.30) by 4π, the area of $\mathbb{S}^2$ and dividing by k, we obtain

$$\mathrm{Area}(\mathbb{S}^2)\ \liminf_{k\to+\infty} \frac{\mathfrak{L}_{1,k}(\mathbb{S}^2)}{k} \geq \pi j_0^2\,. \tag{7.32}$$

But πj_0^2 is the groundstate energy $\lambda(D^1)$ of the Laplacian on the disk D^1 in $\mathbb{R}^2$ of area 1. Although not written explicitly[7] in [23, 8], the Faber-Krahn Inequality gives for planar domains

$$|\Omega|\frac{\mathfrak{L}_{k,1}(\Omega)}{k} \geq \lambda(D^1) = \pi j_0^2\,. \tag{7.33}$$

We have not verified the details, but we think that as in [8] for planar domains, we will have

$$\mathrm{Area}(\mathbb{S}^2)\ \limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\mathbb{S}^2)}{k} \leq \lambda(Hexa^1)\,, \tag{7.34}$$

[7]The authors mention only the lower bound for $\mathfrak{L}_k(\Omega)$

where $Hexa^1$ denotes the regular hexagon of area 1.

As for the case of plane domains, it is natural to conjecture (see for example [8, 12] but we first heard of this question from M. Van den Berg five years ago) that :

Conjecture 7.11

$$\lim_{k\to+\infty}\frac{\mathfrak{L}_k(\mathbb{S}^2)}{k}=\lim_{k\to+\infty}\frac{\mathfrak{L}_{k,1}(\mathbb{S}^2)}{k}=\lambda(Hexa^1)\,.$$

The first equality in the conjecture corresponds to the idea, which is well illustrated in the recent paper by Bourdin-Bucur-Oudet [9] that, asymptotically as $k\to+\infty$, a minimal k-partition for Λ^p will correspond to D_j's such that the $\lambda(D_j)$ are equal.

Remark 7.12
If Ω is a regular bounded open set in $\mathbb{R}^2$ or in $\mathbb{S}^2$, then

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Omega)\leq\mathfrak{L}_{k,1}(\Omega)\,. \tag{7.35}$$

The proof is "fermionic". It is enough to apply the minimax characterization for the groundstate energy $\lambda^{Fermi,k}$ of the Dirichlet realization of the Laplacian on Ω^k (in $(\mathbb{R}^2)^k$ or in $(\mathbb{S}^2)^k$) restricted to the Fermionic space $\wedge^k L^2(\Omega)$ which is

$$\lambda^{Fermi,k}=\sum_{j=1}^{k}\lambda_j(\Omega)\,.$$

For any k-partition $\mathcal{D}$ of Ω, we can indeed consider the Slater determinant of the normalized groundstates ϕ_j of each D_j and observe that the corresponding energy is $k\Lambda^1(\mathcal{D})$.
This suggests the following conjecture (which is proven for $p=1$ and $p=+\infty$) :

$$\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Omega)^p\right)^{\frac{1}{p}}\leq\mathfrak{L}_{k,p}(\Omega)\,,\ \forall k\geq 1\,,\ \forall p\in[1,+\infty]\,. \tag{7.36}$$

The case when Ω is the union of two disks considered in Remark 7.4 gives an example for $k=2$ where (7.35) becomes an equality. In this case, we have indeed $\lambda_1(\Omega)=\lambda(C_1)$ and $\lambda_2(\Omega)=\lambda(C_2)$.

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